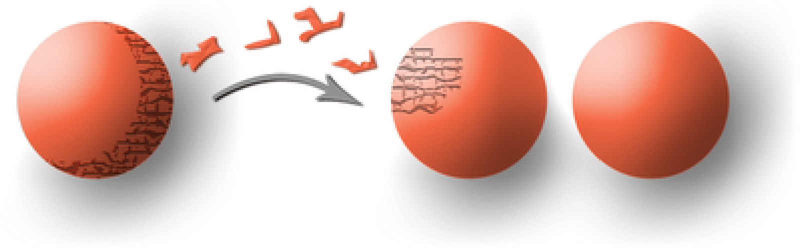
Imagining the Banach-Tarski Paradox

Rachel Levanger

University of North Florida

August 4, 2011

Introduction



The Banach-Tarski Paradox: *A solid ball in 3-dimensional space can be split into a finite number of non-overlapping pieces, which can then be put back together in a different way to yield two identical copies of the original ball.*

[1] Source: Wikipedia

Banach-Tarski in Pop Culture

Futurama June 23, 2011

Banach-Tarski in Pop Culture

XKCD Oct 11, 2010

Definition

Paradoxical

Let X be an infinite set and suppose $E \subseteq X$. We say that E is *paradoxical* if for some positive integers n, m there are pairwise disjoint subsets $A_1, ..., A_n, B_1, ..., B_m$ of E and corresponding permutations $g_1, ..., g_n, h_1, ..., h_m$ of X such that

$$\bigcup g_i(A_i) = E \text{ and } \bigcup h_k(B_k) = E.$$

[2]The formal definition typically given involves that of a group G acting on a set X. To simplify the presentation for an audience of undergraduates, the definition was modified to remove references to group actions, and instead uses sets and permutations. It can be shown that if X is infinite, then it is paradoxical under the group of all permutations.

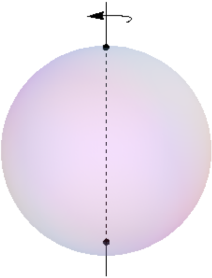
Example: Rotations by $arccos\frac{1}{3} \approx 70.53^\circ$

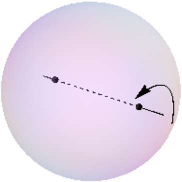
$$\sigma^{\pm 1} = \begin{pmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation by $\arccos\frac{1}{3}$ *around* z*-axis*

$$\tau^{\pm 1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} \\ 0 & \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}$$

Rotation by $\arccos\frac{1}{3}$ *around* x*-axis*





Example: Rotations by $arccos\frac{1}{3} \approx 70.53^\circ$

Why rotate by $\arccos\frac{1}{3}$?

- Angle is an irrational multiple of π
- Iterations of rotations take points to unique images
- Moreover, products of rotations take points to unique images

[3] Fixed points of rotations are addressed later on in the presentation.

A paradoxical set of rotations

F is Paradoxical with Respect to Itself

The set F of finite (reduced) products of the matrices σ, σ^{-1}, τ, and τ^{-1} is paradoxical.

$$\sigma^{\pm 1} = \begin{pmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \tau^{\pm 1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} \\ 0 & \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}$$

Examples:

$\sigma\sigma^{-1} = \text{id}$

$\sigma\sigma$

$\tau\tau\sigma$

$\sigma\tau\tau\sigma^{-1}\tau^{-1}\sigma\sigma$

$\tau^{-1}\sigma\tau\sigma\sigma\tau\sigma^{1}\tau\tau$

$\tau\tau\tau\tau\tau\tau\tau\tau\tau\tau\tau\tau\tau\tau\tau\tau\tau\tau\tau\tau\tau$

A paradoxical set of rotations

$P(\sigma)$	$P(\sigma^{-1})$	$P(\tau)$	$P(\tau^{-1})$
σ	σ^{-1}	τ	τ^{-1}
$\sigma\sigma$	$\sigma^{-1}\sigma^{-1}$	$\tau\tau$	$\tau^{-1}\tau^{-1}$
$\sigma\sigma\ldots$	$\sigma^{-1}\sigma^{-1}\ldots$	$\tau\tau\ldots$	$\tau^{-1}\tau^{-1}\ldots$
$\sigma\tau$	$\sigma^{-1}\tau$	$\tau\sigma$	$\tau^{-1}\sigma$
$\sigma\tau\ldots$	$\sigma^{-1}\tau\ldots$	$\tau\sigma\ldots$	$\tau^{-1}\sigma\ldots$
$\sigma\tau^{-1}$	$\sigma^{-1}\tau^{-1}$	$\tau\sigma^{-1}$	$\tau^{-1}\sigma^{-1}$
$\sigma\tau^{-1}\ldots$	$\sigma^{-1}\tau^{-1}\ldots$	$\tau\sigma^{-1}\ldots$	$\tau^{-1}\sigma^{-1}\ldots$

- Sort products based on left-most rotation in product

A paradoxical set of rotations

$P(\sigma)$	$P(\sigma^{-1})$	$P(\tau)$	$P(\tau^{-1})$
σ	σ^{-1}	τ	τ^{-1}
$\sigma\sigma$	$\sigma^{-1}\sigma^{-1}$	$\tau\tau$	$\tau^{-1}\tau^{-1}$
$\sigma\sigma\ldots$	$\sigma^{-1}\sigma^{-1}\ldots$	$\tau\tau\ldots$	$\tau^{-1}\tau^{-1}\ldots$
$\sigma\tau$	$\sigma^{-1}\tau$	$\tau\sigma$	$\tau^{-1}\sigma$
$\sigma\tau\ldots$	$\sigma^{-1}\tau\ldots$	$\tau\sigma\ldots$	$\tau^{-1}\sigma\ldots$
$\sigma\tau^{-1}$	$\sigma^{-1}\tau^{-1}$	$\tau\sigma^{-1}$	$\tau^{-1}\sigma^{-1}$
$\sigma\tau^{-1}\ldots$	$\sigma^{-1}\tau^{-1}\ldots$	$\tau\sigma^{-1}\ldots$	$\tau^{-1}\sigma^{-1}\ldots$

- Sort products based on left-most rotation in product
- Divide into two groups

A paradoxical set of rotations

$P(\sigma)$	$\sigma P(\sigma^{-1})$	$P(\tau)$	$\tau P(\tau^{-1})$
σ	$\sigma\sigma^{-1}$	τ	$\tau\tau^{-1}$
$\sigma\sigma$	$\sigma\sigma^{-1}\sigma^{-1}$	$\tau\tau$	$\tau\tau^{-1}\tau^{-1}$
$\sigma\sigma\ldots$	$\sigma\sigma^{-1}\sigma^{-1}\ldots$	$\tau\tau\ldots$	$\tau\tau^{-1}\tau^{-1}\ldots$
$\sigma\tau$	$\sigma\sigma^{-1}\tau$	$\tau\sigma$	$\tau\tau^{-1}\sigma$
$\sigma\tau\ldots$	$\sigma\sigma^{-1}\tau\ldots$	$\tau\sigma\ldots$	$\tau\tau^{-1}\sigma\ldots$
$\sigma\tau^{-1}$	$\sigma\sigma^{-1}\tau^{-1}$	$\tau\sigma^{-1}$	$\tau\tau^{-1}\sigma^{-1}$
$\sigma\tau^{-1}\ldots$	$\sigma\sigma^{-1}\tau^{-1}\ldots$	$\tau\sigma^{-1}\ldots$	$\tau\tau^{-1}\sigma^{-1}\ldots$

- Sort products based on left-most rotation in product
- Divide into two groups
- Apply σ and τ rotations

A paradoxical set of rotations

$P(\sigma)$	$\sigma P(\sigma^{-1})$	$P(\tau)$	$\tau P(\tau^{-1})$
σ	id	τ	id
$\sigma\sigma$	σ^{-1}	$\tau\tau$	τ^{-1}
$\sigma\sigma\ldots$	$\sigma^{-1}\ldots$	$\tau\tau\ldots$	$\tau^{-1}\ldots$
$\sigma\tau$	τ	$\tau\sigma$	σ
$\sigma\tau\ldots$	$\tau\ldots$	$\tau\sigma\ldots$	$\sigma\ldots$
$\sigma\tau^{-1}$	τ^{-1}	$\tau\sigma^{-1}$	σ^{-1}
$\sigma\tau^{-1}\ldots$	$\tau^{-1}\ldots$	$\tau\sigma^{-1}\ldots$	$\sigma^{-1}\ldots$

- Sort products based on left-most rotation in product
- Divide into two groups
- Apply σ and τ rotations
- Adjacent rotations cancel to yield paradoxical decomposition

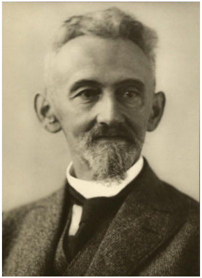
A paradoxical set of rotations

$P(\sigma)$	$\sigma P(\sigma^{-1})$	$P(\tau)$	$\tau P(\tau^{-1})$
σ	id	τ	id
$\sigma\sigma$	σ^{-1}	$\tau\tau$	τ^{-1}
$\sigma\sigma\ldots$	$\sigma^{-1}\ldots$	$\tau\tau\ldots$	$\tau^{-1}\ldots$
$\sigma\tau$	τ	$\tau\sigma$	σ
$\sigma\tau\ldots$	$\tau\ldots$	$\tau\sigma\ldots$	$\sigma\ldots$
$\sigma\tau^{-1}$	τ^{-1}	$\tau\sigma^{-1}$	σ^{-1}
$\sigma\tau^{-1}\ldots$	$\tau^{-1}\ldots$	$\tau\sigma^{-1}\ldots$	$\sigma^{-1}\ldots$

$$P(\sigma) \cup \sigma P(\sigma^{-1}) = F = P(\tau) \cup \tau P(\tau^{-1})$$

- Sort products based on left-most rotation in product
- Divide into two groups
- Apply σ and τ rotations
- Adjacent rotations cancel to yield paradoxical decomposition

The Hausdorff Paradox



Felix Hausdorff

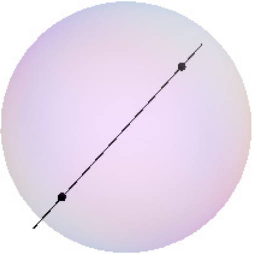
In 1914, Felix Hausdorff finds a way to leverage the paradox on F to a subset of the hollow sphere, S^2.

He originally used a different set of rotations, but the idea is similar to what is presented here.

The Hausdorff Paradox

$S^2 \backslash D$ is Paradoxical *(Felix Hausdorff, 1914)*

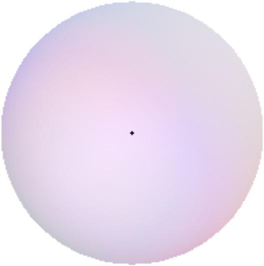
Let D be the set of all fixed points of the hollow sphere S^2 under the rotations in F. The set $S^2 \backslash D$ is paradoxical using four pieces.



Fixed points of a rotation: two points of intersection between the axis of rotation and the sphere

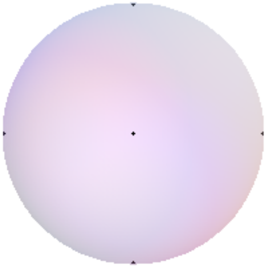
The Hausdorff Paradox

First we look at the image of a single point under rotations in F, the finite reduced products of σ, σ^{-1}, τ, and τ^{-1}:



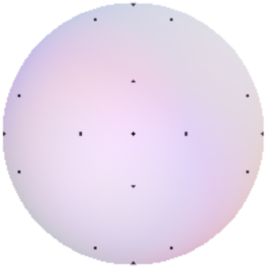
The Hausdorff Paradox

First we look at the image of a single point under rotations in F, the finite reduced products of σ, σ^{-1}, τ, and τ^{-1}:



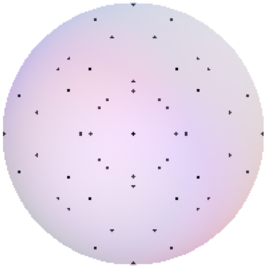
The Hausdorff Paradox

First we look at the image of a single point under rotations in F, the finite reduced products of σ, σ^{-1}, τ, and τ^{-1}:



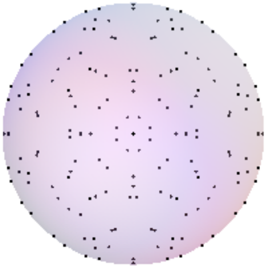
The Hausdorff Paradox

First we look at the image of a single point under rotations in F, the finite reduced products of σ, σ^{-1}, τ, and τ^{-1}:

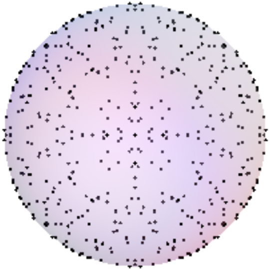


The Hausdorff Paradox

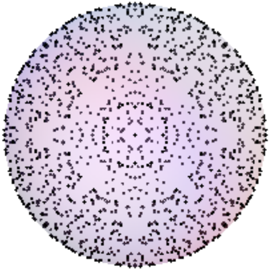
First we look at the image of a single point under rotations in F, the finite reduced products of σ, σ^{-1}, τ, and τ^{-1}:



First we look at the image of a single point under rotations in F, the finite reduced products of σ, σ^{-1}, τ, and τ^{-1}:

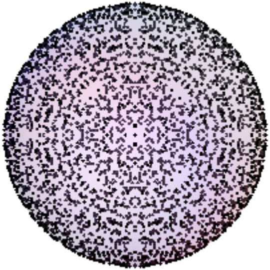


First we look at the image of a single point under rotations in F, the finite reduced products of σ, σ^{-1}, τ, and τ^{-1}:



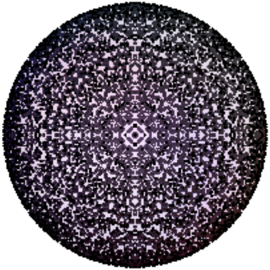
The Hausdorff Paradox

First we look at the image of a single point under rotations in F, the finite reduced products of σ, σ^{-1}, τ, and τ^{-1}:



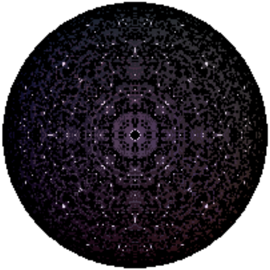
The Hausdorff Paradox

First we look at the image of a single point under rotations in F, the finite reduced products of σ, σ^{-1}, τ, and τ^{-1}:



The Hausdorff Paradox

First we look at the image of a single point under rotations in F, the finite reduced products of σ, σ^{-1}, τ, and τ^{-1}:



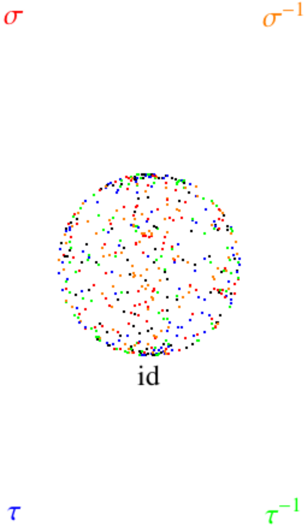
The Hausdorff Paradox

How do we create a correspondence between a partition of $S^2 \backslash D$ and F?

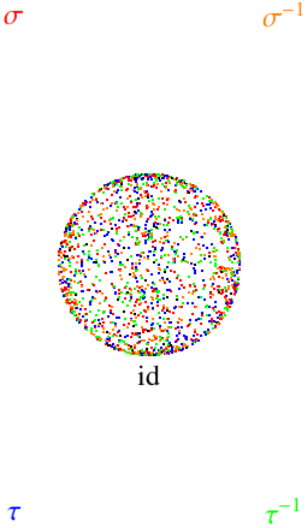
- Each image set is countable, since F is
- An uncountable number of these image sets partition $S^2 \backslash D$
- Using the Axiom of Choice, we create a choice set by choosing one representative from each image set in the partition
- The choice set is uncountable and, when rotated by elements of F, creates the correspondence we're looking for

The Hausdorff Paradox

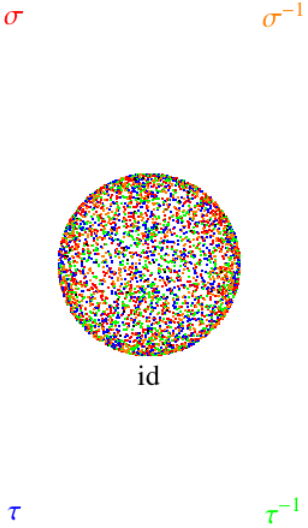
The Hausdorff Paradox



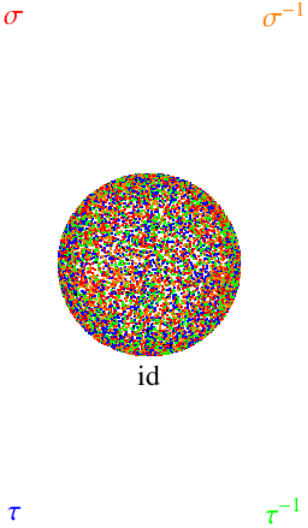
The Hausdorff Paradox



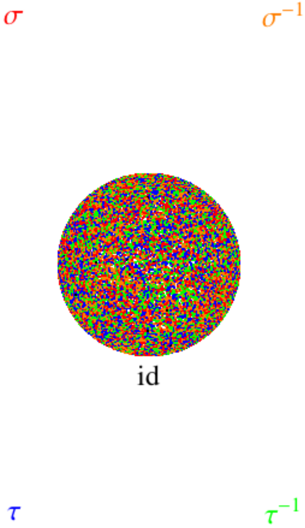
The Hausdorff Paradox

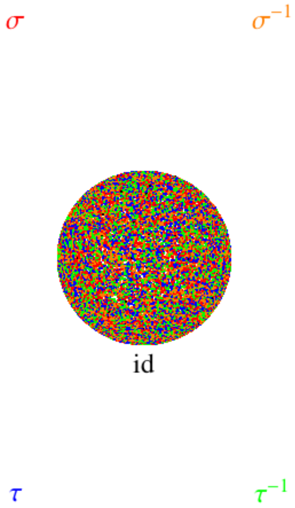


The Hausdorff Paradox



The Hausdorff Paradox




σ
σ⁻¹
id
τ
τ⁻¹

Piecewise Congruence

Piecewise congruent, $A \sim B$

Suppose $A, B \subseteq X$. Then A and B are *piecewise congruent* if for some positive integer n, there exists a

- partition of A, $\{A_i : 1 \leq i \leq n\}$
- partition of B, $\{B_i : 1 \leq i \leq n\}$
- set of rigid motions $g_1, ..., g_n$ of X

such that $g_i(A_i) = B_i$ for each $1 \leq i \leq n$. We write that $A \sim B$ if such a correspondence exists.

Piecewise congruence preserves paradoxes

If $A \sim B$ and A is paradoxical, then B is paradoxical.

[4] This definition is also referred to as *equidecomposable*.

"Filling in a Point"

$S^1 \sim S^1 \backslash (1, 0)$

The unit circle S^1 is piecewise congruent to $S^1 \backslash (1, 0)$.

“Filling in a Point”

$S^1 \sim S^1 \backslash (1,0)$

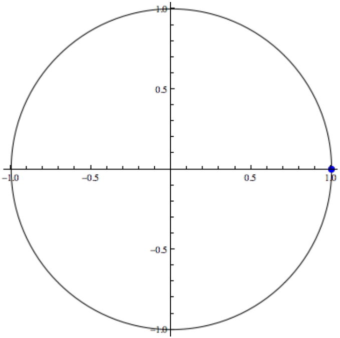
The unit circle S^1 is piecewise congruent to $S^1 \backslash (1,0)$.

Create partition set:

"Filling in a Point"

$S^1 \sim S^1 \backslash (1,0)$

The unit circle S^1 is piecewise congruent to $S^1 \backslash (1,0)$.



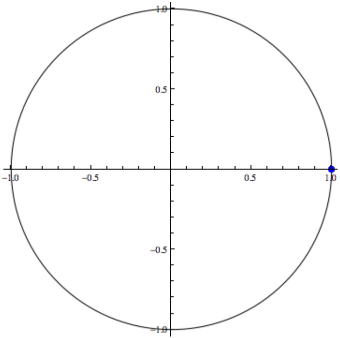
Create partition set:

1. Isolate the "hole"

"Filling in a Point"

$S^1 \sim S^1 \backslash (1,0)$

The unit circle S^1 is piecewise congruent to $S^1 \backslash (1,0)$.



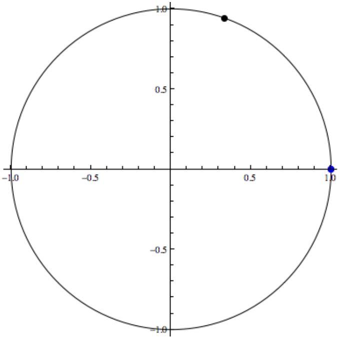
Create partition set:

1. Isolate the "hole"
2. Repeatedly apply rotation $\phi = arccos\frac{1}{3}$ to the hole

"Filling in a Point"

$S^1 \sim S^1 \backslash (1,0)$

The unit circle S^1 is piecewise congruent to $S^1 \backslash (1,0)$.



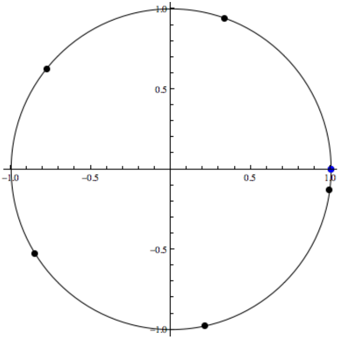
Create partition set:

1. Isolate the "hole"
2. Repeatedly apply rotation $\phi = arccos\frac{1}{3}$ to the hole

"Filling in a Point"

$S^1 \sim S^1 \backslash (1,0)$

The unit circle S^1 is piecewise congruent to $S^1 \backslash (1,0)$.



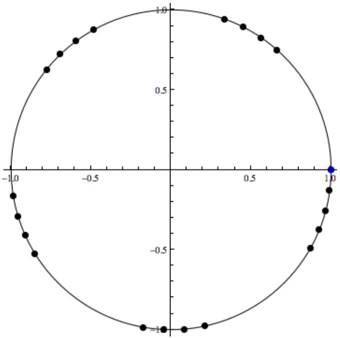
Create partition set:

1. Isolate the "hole"
2. Repeatedly apply rotation $\phi = arccos\frac{1}{3}$ to the hole

"Filling in a Point"

$S^1 \sim S^1 \backslash (1,0)$

The unit circle S^1 is piecewise congruent to $S^1 \backslash (1,0)$.



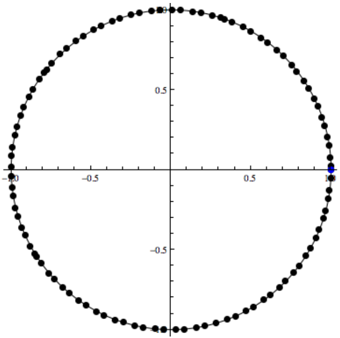
Create partition set:

1. Isolate the "hole"
2. Repeatedly apply rotation $\phi = arccos\frac{1}{3}$ to the hole

"Filling in a Point"

$S^1 \sim S^1 \backslash (1,0)$

The unit circle S^1 is piecewise congruent to $S^1 \backslash (1,0)$.



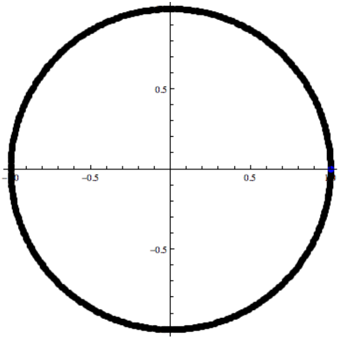
Create partition set:

1. Isolate the "hole"
2. Repeatedly apply rotation $\phi = arccos\frac{1}{3}$ to the hole

"Filling in a Point"

$S^1 \sim S^1 \backslash (1,0)$

The unit circle S^1 is piecewise congruent to $S^1 \backslash (1,0)$.



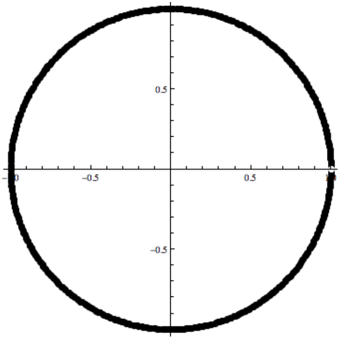
Create partition set:

1. Isolate the "hole"
2. Repeatedly apply rotation $\phi = arccos \frac{1}{3}$ to the hole

"Filling in a Point"

$S^1 \sim S^1 \backslash (1,0)$

The unit circle S^1 is piecewise congruent to $S^1 \backslash (1,0)$.



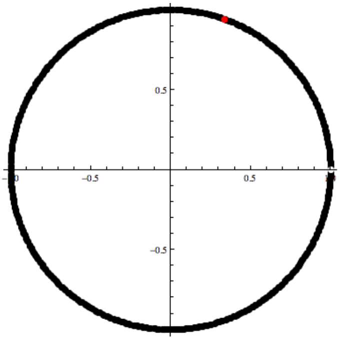
Create partition set:

1. Isolate the "hole"
2. Repeatedly apply rotation $\phi = arccos\frac{1}{3}$ to the hole
3. Look only at the image set of the hole, $\overline{(1,0)}$

"Filling in a Point"

$S^1 \sim S^1 \backslash (1,0)$

The unit circle S^1 is piecewise congruent to $S^1 \backslash (1,0)$.



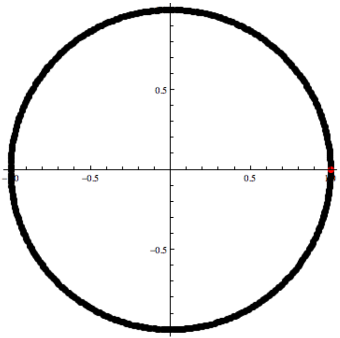
Create partition set:

1. Isolate the "hole"
2. Repeatedly apply rotation $\phi = arccos\frac{1}{3}$ to the hole
3. Look only at the image set of the hole, $\overline{(1,0)}$
4. Apply inverse rotation, ϕ^{-1}

“Filling in a Point”

$S^1 \sim S^1 \backslash (1,0)$

The unit circle S^1 is piecewise congruent to $S^1 \backslash (1,0)$.



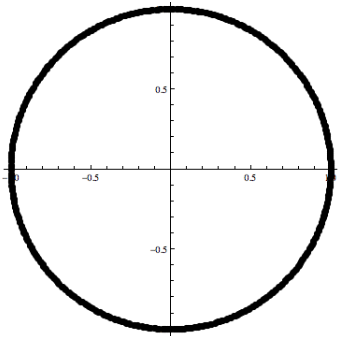
Create partition set:

1. Isolate the “hole”
2. Repeatedly apply rotation $\phi = arccos\frac{1}{3}$ to the hole
3. Look only at the image set of the hole, $\overline{(1,0)}$
4. Apply inverse rotation, ϕ^{-1}

"Filling in a Point"

$S^1 \sim S^1 \backslash (1,0)$

The unit circle S^1 is piecewise congruent to $S^1 \backslash (1,0)$.



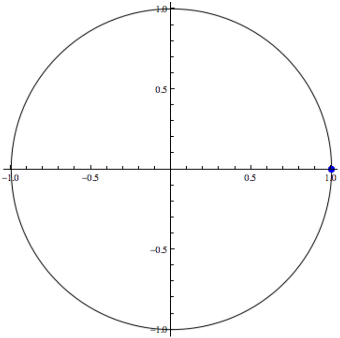
Create partition set:

1. Isolate the "hole"
2. Repeatedly apply rotation $\phi = arccos\frac{1}{3}$ to the hole
3. Look only at the image set of the hole, $\overline{(1,0)}$
4. Apply inverse rotation, ϕ^{-1}

"Filling in a Point"

$S^1 \sim S^1 \backslash (1,0)$

The unit circle S^1 is piecewise congruent to $S^1 \backslash (1,0)$.

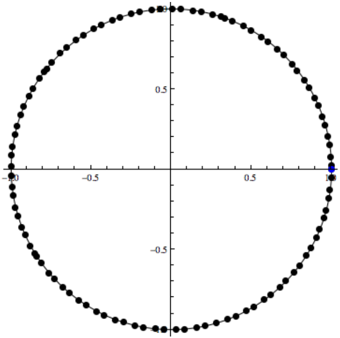


Piecewise congruent partition:

"Filling in a Point"

$S^1 \sim S^1 \backslash (1,0)$

The unit circle S^1 is piecewise congruent to $S^1 \backslash (1,0)$.



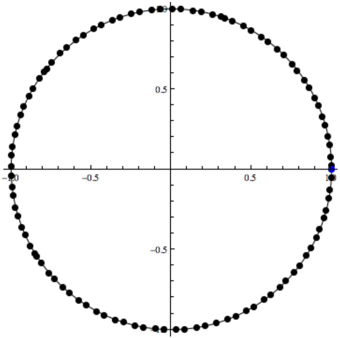
Piecewise congruent partition:

- The image set, $\overline{(1,0)}$

"Filling in a Point"

$S^1 \sim S^1 \backslash (1,0)$

The unit circle S^1 is piecewise congruent to $S^1 \backslash (1,0)$.



Piecewise congruent partition:

- The image set, $\overline{(1,0)}$
- The rest of the points, $S^1 \backslash \overline{(1,0)}$

"Filling in the set D"

$S^2 \sim S^2 \backslash D$

The unit sphere S^2 is piecewise congruent to $S^2 \backslash D$.

"Filling in the set D"

$S^2 \sim S^2 \backslash D$

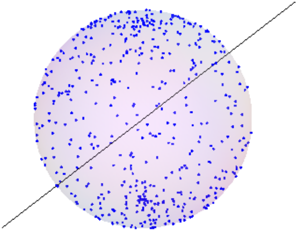
The unit sphere S^2 is piecewise congruent to $S^2 \backslash D$.

Create partition set:

"Filling in the set D"

$S^2 \sim S^2 \setminus D$

The unit sphere S^2 is piecewise congruent to $S^2 \setminus D$.



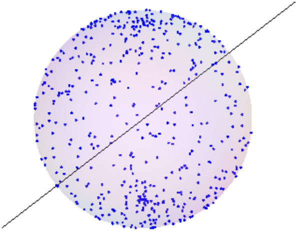
Create partition set:

1. Isolate the "holes," D

"Filling in the set D"

$S^2 \sim S^2 \backslash D$

The unit sphere S^2 is piecewise congruent to $S^2 \backslash D$.



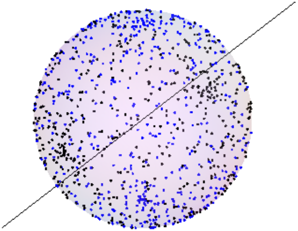
Create partition set:

1. Isolate the "holes," D
2. Repeatedly apply rotation ϕ to the holes

"Filling in the set D"

$S^2 \sim S^2 \setminus D$

The unit sphere S^2 is piecewise congruent to $S^2 \setminus D$.



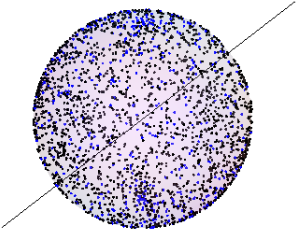
Create partition set:

1. Isolate the "holes," D
2. Repeatedly apply rotation ϕ to the holes

"Filling in the set D"

$S^2 \sim S^2 \backslash D$

The unit sphere S^2 is piecewise congruent to $S^2 \backslash D$.



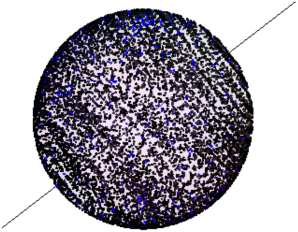
Create partition set:

1. Isolate the "holes," D
2. Repeatedly apply rotation ϕ to the holes

"Filling in the set D"

$S^2 \sim S^2 \setminus D$

The unit sphere S^2 is piecewise congruent to $S^2 \setminus D$.



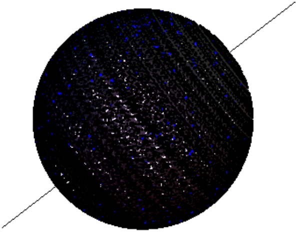
Create partition set:

1. Isolate the "holes," D
2. Repeatedly apply rotation ϕ to the holes

"Filling in the set D"

$S^2 \sim S^2 \setminus D$

The unit sphere S^2 is piecewise congruent to $S^2 \setminus D$.



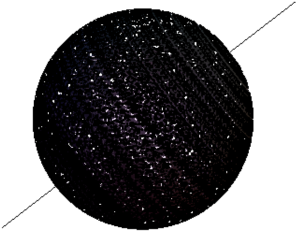
Create partition set:

1. Isolate the "holes," D
2. Repeatedly apply rotation ϕ to the holes

"Filling in the set D"

$S^2 \sim S^2 \setminus D$

The unit sphere S^2 is piecewise congruent to $S^2 \setminus D$.



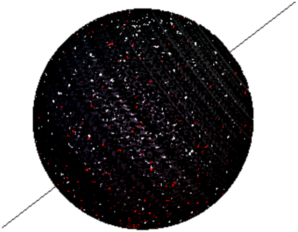
Create partition set:

1. Isolate the "holes," D
2. Repeatedly apply rotation ϕ to the holes
3. Look only at the image set of the hole, $\overline{D}$

"Filling in the set D"

$S^2 \sim S^2 \setminus D$

The unit sphere S^2 is piecewise congruent to $S^2 \setminus D$.



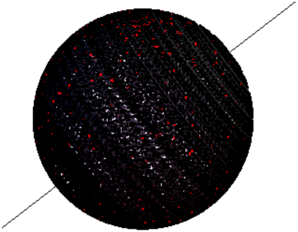
Create partition set:

1. Isolate the "holes," D
2. Repeatedly apply rotation ϕ to the holes
3. Look only at the image set of the hole, $\overline{D}$
4. Apply inverse rotation, ϕ^{-1}

"Filling in the set D"

$S^2 \sim S^2 \backslash D$

The unit sphere S^2 is piecewise congruent to $S^2 \backslash D$.



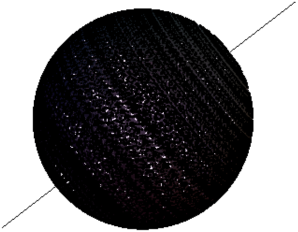
Create partition set:

1. Isolate the "holes," D
2. Repeatedly apply rotation ϕ to the holes
3. Look only at the image set of the hole, $\overline{D}$
4. Apply inverse rotation, ϕ^{-1}

"Filling in the set D"

$S^2 \sim S^2 \backslash D$

The unit sphere S^2 is piecewise congruent to $S^2 \backslash D$.



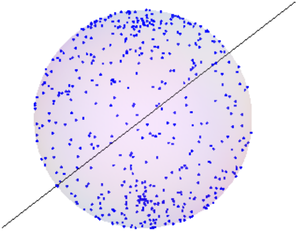
Create partition set:

1. Isolate the "holes," D
2. Repeatedly apply rotation ϕ to the holes
3. Look only at the image set of the hole, $\overline{D}$
4. Apply inverse rotation, ϕ^{-1}

“Filling in the set D”

$S^2 \sim S^2 \backslash D$

The unit sphere S^2 is piecewise congruent to $S^2 \backslash D$.

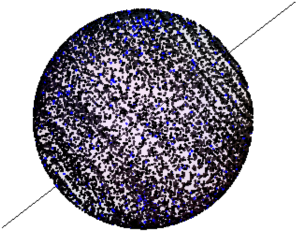


Piecewise congruent partition:

"Filling in the set D"

$S^2 \sim S^2 \backslash D$

The unit sphere S^2 is piecewise congruent to $S^2 \backslash D$.



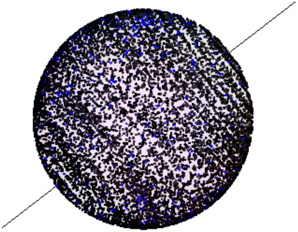
Piecewise congruent partition:

- The image set, $\overline{D}$

"Filling in the set D"

$S^2 \sim S^2 \setminus D$

The unit sphere S^2 is piecewise congruent to $S^2 \setminus D$.



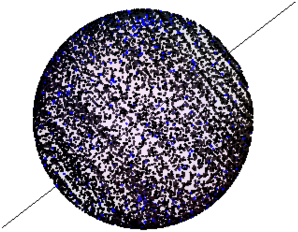
Piecewise congruent partition:

- The image set, $\overline{D}$
- The rest of the points, $S^2 \setminus \overline{D}$

"Filling in the set D"

$S^2 \sim S^2 \backslash D$

The unit sphere S^2 is piecewise congruent to $S^2 \backslash D$.

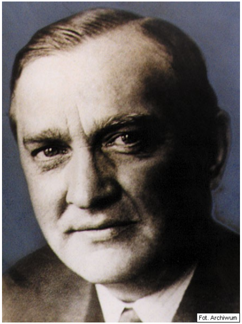


Piecewise congruent partition:

- The image set, $\overline{D}$
- The rest of the points, $S^2 \backslash \overline{D}$

By the Hausdorff Paradox,
and since piecewise congruence
preserves paradoxical
decompositions,
S^2 is Paradoxical!

The Banach-Tarski Paradox



Stefan Banach

Alfred Tarski

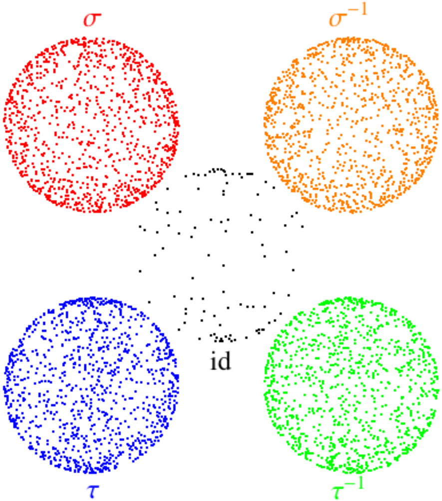
The Banach-Tarski Paradox

$B \backslash (0,0,0)$ is paradoxical

The unit ball minus the origin $B \backslash (0,0,0)$ is paradoxical with respect to rotations in $\mathbb{R}^3$.

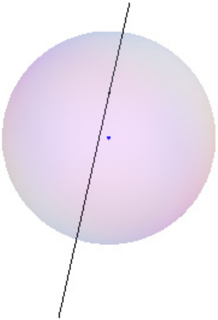
- Begin with the Hausdorff Paradox decomposition
- Use radial correspondence between S^2 and $B \backslash (0,0,0)$
- Apply the same rotations needed to create paradox with S^2

The Banach-Tarski Paradox



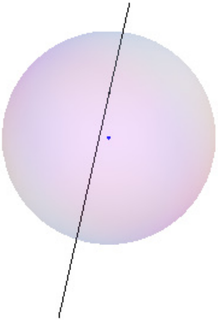
$B\backslash(0,0,0) \sim B$

The unit ball minus origin $B\backslash(0,0,0)$ is piecewise congruent to B with respect to isometries of $\mathbb{R}^3$.



$B\backslash(0,0,0) \sim B$

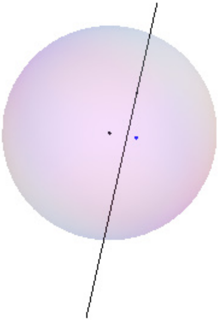
The unit ball minus origin $B\backslash(0,0,0)$ is piecewise congruent to B with respect to isometries of $\mathbb{R}^3$.



1. Isolate the "hole"

$B \backslash (0,0,0) \sim B$

The unit ball minus origin $B \backslash (0,0,0)$ is piecewise congruent to B with respect to isometries of $\mathbb{R}^3$.

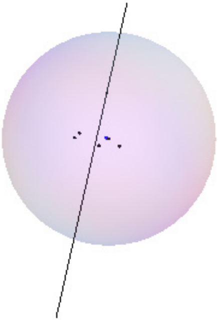


1. Isolate the "hole"
2. Repeatedly apply rotation ϕ around ℓ to the hole

The Banach-Tarski Paradox

$B \backslash (0,0,0) \sim B$

The unit ball minus origin $B \backslash (0,0,0)$ is piecewise congruent to B with respect to isometries of $\mathbb{R}^3$.

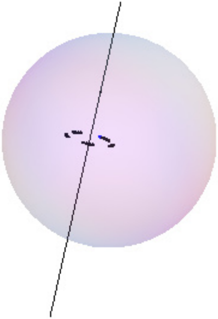


1. Isolate the "hole"
2. Repeatedly apply rotation ϕ around ℓ to the hole

The Banach-Tarski Paradox

$B \backslash (0,0,0) \sim B$

The unit ball minus origin $B \backslash (0,0,0)$ is piecewise congruent to B with respect to isometries of $\mathbb{R}^3$.

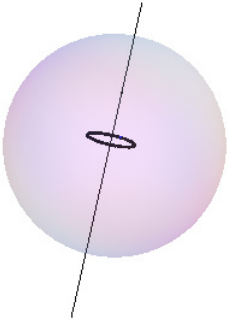


1. Isolate the "hole"
2. Repeatedly apply rotation ϕ around ℓ to the hole

The Banach-Tarski Paradox

$B \backslash (0,0,0) \sim B$

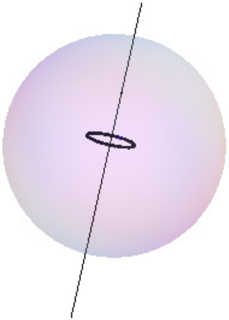
The unit ball minus origin $B \backslash (0,0,0)$ is piecewise congruent to B with respect to isometries of $\mathbb{R}^3$.



1. Isolate the "hole"
2. Repeatedly apply rotation ϕ around ℓ to the hole

$B \backslash (0,0,0) \sim B$

The unit ball minus origin $B \backslash (0,0,0)$ is piecewise congruent to B with respect to isometries of $\mathbb{R}^3$.

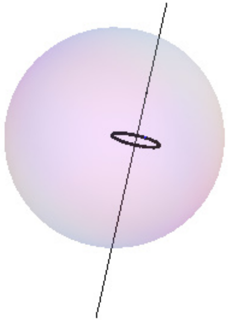


1. Isolate the "hole"
2. Repeatedly apply rotation ϕ around ℓ to the hole
3. Look only at the image set of the hole, $\overline{(0,0,0)}$

The Banach-Tarski Paradox

$B \setminus (0,0,0) \sim B$

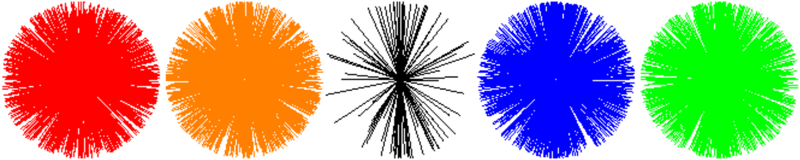
The unit ball minus origin $B \setminus (0,0,0)$ is piecewise congruent to B with respect to isometries of $\mathbb{R}^3$.



1. Isolate the "hole"
2. Repeatedly apply rotation ϕ around ℓ to the hole
3. Look only at the image set of the hole, $\overline{(0,0,0)}$
4. Apply inverse rotation, ϕ^{-1} to fill in the holes

The Banach-Tarski Paradox *(Banach and Tarski, 1935)*

The solid unit ball centered at the origin in $\mathbb{R}^3$ is paradoxical using isometries on $\mathbb{R}^3$.



References

S. Wagon, *The Banach-Tarski Paradox*, Cambridge Univ. Press, Cambridge, 1993, pp. 1–28.