Imagining the Banach-Tarski Paradox

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The contents of this exposition are primarily due to S. Wagon's book entitled "The Banach-Tarski Paradox." I have attempted to provide an abbreviated form of the first three chapters, focusing on content relevant to the most commonly stated form of the Banach-Tarski Paradox and reformulating many results in more accessible terms. I have also used a series of examples traced throughout, providing diagrams where possible to enhance intuition about the construction of the paradox and the nature of the sets used to exhibit such counterintuitive behavior.

This exposition is intended to be read by undergraduates who have had a course in linear algebra and who have been introduced to notions of cardinality in our real number system, equivalence classes, and the content presented in a first course in abstract algebra.

1 Introduction

The Banach-Tarski Paradox is commonly presented as follows: Beginning with a solid sphere in $\mathbb{R}^3$, one can partition the sphere into a finite number of disjoint subsets and, applying only rigid distance-preserving transformations to each, form two spheres of original size and volume to the first. To non-mathematicians, the description goes something like: Beginning with a solid ball, one can break the ball into five pieces, rotate and move the pieces (without distorting or stretching), and then recombine the pieces to form two balls of original size and volume to the first, essentially creating two balls from one.

At first, either description of the Banach-Tarski Paradox seems utterly counterintuitive. However, once familiarity of paradoxical decompositions is attained through working with more elementary constructions and tracing the development of the proof of the Banach-Tarski Paradox through a series of examples, what once seemed dubious comes to seem entirely plausible. We will begin by looking at paradoxical compositions of the integers, then develop paradoxes involving the sphere and eventually the solid ball in $\mathbb{R}^3$.

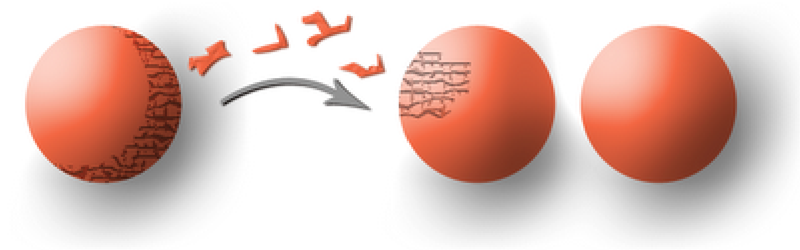


Figure 1: Wikipedia Illustration of the Banach-Tarski Paradox

2 Paradoxical Decompositions

Paradoxical constructions have intrigued mathematicians since Galileo. In fact, it was Galileo who noticed that the square integers could be put into a one-to-one correspondence with the set of all positive integers, even though intuition tells us the latter is far more numerous than the former. Galileo also noticed a type of duplicative paradox, where it is possible to take the set of all positive integers, divide them into two disjoint subsets, and show that these sets are the same size as the original set. Going even farther, we can show that it is possible to take the set of all integers, divide them into two disjoint subsets, perform bijective operations on the subsets individually, and get back two copies of the integers.

This leads us to our first definition. As we are looking to use bijections to form our paradoxical decompositions (and eventually distance-preserving bijections), the most natural way to do this is to look to permutations. Recall that a set, G, of permutations is called a group under function composition if and only if:

1. the composition of any two permutations in G is still in G;
2. composition of permutations is associative;
3. the identity permutation, 1, is in G; and
4. for every permutation in G, its inverse is also in G.

We are now ready to formally introduce the idea of a paradoxical decomposition. An example will follow to clarify the notation.

Definition 2.1. Let G be a group of permutations of a set X and suppose $E \subseteq X$. We say that E is G-*paradoxical* (or paradoxical with respect to G) if for some positive integers m, n there are pairwise disjoint subsets $A_1, ..., A_n, B_1, ..., B_m$ of E and $g_1, ..., g_n, h_1, ..., h_m \in G$ such that $E = \bigcup g_i(A_i)$ and $E = \bigcup h_j(B_j)$.

Notice that the permutations are defined in terms of X, and not in terms of E directly. This may be an important distinction, as our first example shows.

Example 2.2. *The integers are paradoxical with respect to the group of linear functions with positive slope.*

Consider the integers, $\mathbb{Z}$, as a subset of the real numbers, $\mathbb{R}$. Let G be the group of all linear functions $f : \mathbb{R} \to \mathbb{R}$ with positive slope. (It is easy to check that G is a group of permutations of $\mathbb{R}$ under function composition.) Partition $\mathbb{Z}$ into two disjoint subsets, the even and the odd integers, given by $2\mathbb{Z}$ and $2\mathbb{Z}+1$. Associate with the even integers the bijection given by $g(x) = x/2$, and associate to the odd integers the bijection $h(x) = (x-1)/2$. Then $\mathbb{Z} = g(2\mathbb{Z})$ and $\mathbb{Z} = h(2\mathbb{Z}+1)$, rendering $\mathbb{Z}$ paradoxical with respect to G. Since we have two subsets in the partition and each element of G acts on a subset individually, we have created this decomposition with two pieces. □

In other words, we have partitioned $\mathbb{Z}$ into two disjoint subsets and performed bijective operations on each to recreate two identical copies of $\mathbb{Z}$! What this example exploits is the fact that the integers are infinite (and unbounded); we are able to reach out farther and father into infinity (twice as far as elements in the original set) to create our correspondence. In this case, we were allowed to use permutations that compress the original set, $\mathbb{R}$, in order to create the correspondence. In the Banach-Tarski Paradox, however, we are given only distance-preserving transformations, such as rotations and translations.

Our next example, which will be used as a foundation in developing the Banach-Tarski Paradox, illustrates a paradoxical decomposition of such transformations. In this case, we are looking only at a group of rotations around a line through the origin, which are considered distance-preserving permutations in $\mathbb{R}^3$. We give a definition to formally introduce notation that will be used throughout the paper, and then proceed with our example, which is ultimately the basis of the Banach-Tarski Paradox.

Definition 2.3. We will denote the group of rotations about lines through the origin in $\mathbb{R}^3$ by SO_3, after the *special orthogonal group*, the group of 3×3 orthogonal matrices with determinant equal to 1.

Definition 2.4. A finite product of permutations σ, σ^{-1}, τ, τ^{-1} is called *reduced* if an element and its inverse do not appear as adjacent terms anywhere in the product.

Proposition 2.5. There are two rotations, σ and τ, about axes through the origin in $\mathbb{R}^3$ that, together with their inverse rotations, generate a subgroup F of rotations of SO_3 such that every finite, reduced product of elements in $\sigma^{\pm 1}$, $\tau^{\pm 1}$ is unique.

Proof. Let σ and τ be counterclockwise rotations around the z-axis and x-axis, respectively, each through the angle $\arccos \frac{1}{3}$. Then $\sigma^{\pm 1}$, $\tau^{\pm 1}$ are represented by the following rotation matrices:

$$\sigma^{\pm 1} = \begin{pmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \tau^{\pm 1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} \\ 0 & \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}.$$

First, we wish to prove that no nontrivial, reduced, finite product in $\sigma^{\pm 1}$, $\tau^{\pm 1}$ equals the identity. For if this is the case, then every nontrivial, reduced, finite product is distinct. Indeed, suppose that w and v are such distinct products and that $wv^{-1} = 1$. Then we would have $w = v$, which is exactly what we are trying to disprove. Without loss of generality, assume that w is a reduced, finite product that ends, on the right, with $\sigma^{\pm 1}$ but that equals the identity. We will thus argue by contradiction.

We claim that since $w(1,0,0)$ is a finite product of $\sigma^{\pm 1}$, $\tau^{\pm 1}$, it has the form $(a, b\sqrt{2}, c)/3^k$ where a, b, c are integers and b is not divisible by 3. However, this would imply that $w(1,0,0) \neq (1,0,0)$, which is the required contradiction. The claim is proved by induction on the length of w. If w has length one, then $w = \sigma^{\pm 1}$ and $w(1,0,0) = (1, \pm 2\sqrt{2}, 0)/3$. Suppose then that $w = \sigma^{\pm 1} w'$ or $w = \tau^{\pm 1} w'$ where $w'(1,0,0) = (a', b'\sqrt{2}, c')/3^{k-1}$. A single application of each of the matrices above yields that $w(1,0,0) = (a, b\sqrt{2}, c)/3^k$ where $a = a' \pm 4b'$, $b = b' \pm 2a'$, $c = 3c'$, or $a = 3a'$, $b = b' \mp 2c'$, $c = c' \pm 4b'$ depending on whether w begins with $\sigma^{\pm 1}$ or $\tau^{\pm 1}$ respectively. It follows that a, b, c are integers by the induction hypothesis (since a', b', c' are integers).

It remains to show that b is never divisible by 3. There are four cases to examine, according to whether w is equal to $\sigma^{\pm 1}\tau^{\pm 1}v$, $\tau^{\pm 1}\sigma^{\pm 1}v$, $\sigma^{\pm 1}\sigma^{\pm 1}v$, or $\tau^{\pm 1}\tau^{\pm 1}v$ where v is a finite product in $\sigma^{\pm 1}$, $\tau^{\pm 1}$, or possibly the identity. In the first two cases, using the reasoning from the previous paragraph, we have that $b = b' \mp 2c'$ where 3 divides c' or $b = b' \pm 2a'$ where 3 divides a'. Thus if b' is not divisible by 3, neither is b. For the other two cases, let a'', b'', c'' be the integers arising in $v(1,0,0)$. Then in either case, $b = 2b' - 9b''$. For instance, in the third case, $b = b' \pm 2a' = b' \pm 2(a'' \mp 4b'') = b' + b'' \pm 2a'' - 9b'' = 2b' - 9b''$; an essentially identical proof works in the fourth case. Thus if b' is not divisible by 3, neither is b, completing the proof.

To show that the set of reduced, finite products in $\sigma^{\pm 1}$, $\tau^{\pm 1}$ forms a subgroup of SO_3, first note that the identity is accounted for by $\sigma\sigma^{-1}$ or a similar product. Thus, F is nonempty. Secondly, given any finite products w and v in $\sigma^{\pm 1}$, $\tau^{\pm 1}$, we see that v^{-1} is also a finite product in $\sigma^{\pm 1}$, $\tau^{\pm 1}$. Thus, wv^{-1} is also a finite product in $\sigma^{\pm 1}$, $\tau^{\pm 1}$, and so $wv^{-1} \in F$. Hence, F is a subgroup of SO_3. □

□

Theorem 2.6. *The subgroup F in Proposition 2.5 is paradoxical with respect to itself.*

Proof. Let F be the subgroup defined in Proposition 2.5. Let ρ be one of $\sigma^{\pm 1}$, $\tau^{\pm 1}$ and denote $W(\rho)$ as the set of elements of F whose representation as a finite, reduced product in $\sigma^{\pm 1}$, $\tau^{\pm 1}$ begins, on the left, with ρ. Then $F = 1 \cup W(\sigma) \cup W(\sigma^{-1}) \cup W(\tau) \cup W(\tau^{-1})$, and these subsets are pairwise disjoint. Furthermore, $W(\sigma) \cup \sigma W(\sigma^{-1}) = F$ and $W(\tau) \cup \tau W(\tau^{-1}) = F$, since if $h \in F \backslash W(\sigma)$, then $\sigma^{-1}h \in W(\sigma^{-1})$ and $h = \sigma(\sigma^{-1}h) \in \sigma W(\sigma^{-1})$. Hence, F is paradoxical with respect to itself, using only four pieces. □

We have now identified the mechanism by which the Banach-Tarski Paradox will be realized. That is to say, we have found a particular subgroup of SO_3 that is paradoxical with respect to itself. Now, we must find a way to leverage this paradoxical group to render a subset of $\mathbb{R}^3$ paradoxical. The following development accomplishes such a task.

3 The Hausdorff Paradox

In 1914, Felix Hausdorff gave the first example of a subgroup of SO_3 that behaves as the subgroup F in Proposition 2.5. Various mathematicians improved on his construction, and in 1958, Świerczkowski provided the construction given. The goal, however, did not stop at finding such a subgroup. Ultimately, the idea was to determine if the unit sphere, S^2, was paradoxical under such a subgroup. While Hausdorff did not determine such a fact, he did pave the way by finding a subset of S^2 that is F-paradoxical. Before we get to the detailed aspects of our development, we will briefly discuss the role of the Axiom of Choice in the construction of the Hausdorff and Banach-Tarski Paradox.

Definition 3.1. **(The Axiom of Choice)** Let $\mathcal{C}$ be a collection of non-empty sets. Then it is possible to create a new set, M, that contains exactly one element from each set in $\mathcal{C}$. Such a set is called a *choice set.*

While the Axiom of Choice seems natural enough to the mathematical mind, it is precisely this axiom that enables paradoxical decompositions in $\mathbb{R}^3$. Without it, it has been shown that the Banach-Tarski Paradox is not possible. The primary objective in defining a paradoxical decomposition of a set is determining the correct collection of pairwise disjoint subsets that will result in the paradox. Beginning with a set X and a paradoxical group G acting on that set (in our case S^2 and the subgroup F), our goal is to use the paradoxical nature of the group to inform an appropriate decomposition of X; essentially, we need to find a decomposition of X that aligns with the paradoxical decomposition of G. As we will show, there is a natural partition of X that is imposed by G. We then use the Axiom of Choice and choose an element from each set in this natural partition. This choice set, M, behaves in an interesting way when certain conditions on X and G are met, leading to the desired paradoxical decomposition. However, these conditions require that F acts not on the whole of S^2, but a subset of S^2 relating to F in a particular way. A few definitions will set the stage for our development.

Definition 3.2. Let G be a group of permutations of a set X. An element $x \in X$ is said to be a *nontrivial fixed point* if there exists some non-identity permutation $g \in G$ where $g(x) = x$. If no such points exist, then G is said to act on X without nontrivial fixed points.

Definition 3.3. Let G be a group of permutations of a set X. Then given $x \in X$, the *orbit of x under G* (or the *G-orbit of x*) is the set $\{g(x) : g \in G\}$, and is denoted by $G(x)$.

Proposition 3.4. *Let G be a group of permutations of a set X. Then the collection of orbits of elements of X under G forms a partition of X, together called the G-orbits of X.*

Proof. First, each element $x \in X$ is an element of $G(x)$, and so the union of all orbits is X. To show that the orbits are disjoint, suppose first that $G(x)$ is an orbit and $y \in G(x)$. Then $y = g(x) \in G(x)$ for some $g \in G$ iff $x = g^{-1}(y) \in G(y)$. Thus, we have shown that $G(x) = G(y)$ if they are not disjoint. Hence, the distinct orbits under G in X form a partition of X. □

As mentioned in the introduction to this section, the natural partition of X that comes from G is the partition of X into G-orbits. This, in conjunction with the requirement that G acts on X without nontrivial fixed points, is all we need to create a choice set with the property we seek, which yields the decomposition leading to the Hausdorff Paradox.

Lemma 3.5. *Let G be a group of permutations of a set X acting without nontrivial fixed points and M a choice set created by choosing a single element from each G-orbit in X. Then the collection of sets $\{g(M) : g \in G\}$ forms a partition of X.*

Proof. To show that every $x \in X$ is contained in at least one set in the collection, note that M contains exactly one element from $G(x)$, the G-orbit containing x (there is only one such G-orbit by Proposition 3.4). Denote this element by $g(x)$. Since $g^{-1} \in G$, then $x = g^{-1}(g(x)) \in g^{-1}(M)$. Hence, x is in at least one set in the collection. To show that the sets are pairwise disjoint, consider $g, h \in G$ where $g \neq h$. Suppose $x \in g(M) \cap h(M)$. Then there exist elements $p, q \in M$ such that $g(p) = x = h(q)$. Since the orbits of p and q are distinct if $p \neq q$ by the way M was defined, we must have that $p = q$. Thus, $g(p) = h(p)$ and hence $p = g^{-1}(h(p)) = (g^{-1}h)(p)$. But this cannot be since $g^{-1}h$ is not the identity permutation and G acts on X without nontrivial fixed points. Hence, the sets in $\{g(M) : g \in G\}$ are disjoint and form a partition of X. □

Hence, we have found a mechanism by which to correspond sets belonging to a partition of X to elements in G. We are now ready to show this results in our ability to paradoxically decompose X with respect to a paradoxical group G, provided that G acts on X without nontrivial fixed points. Indeed, with the work we have shown, using the paradoxical decomposition of G to determine a similar decomposition of X follows quite naturally.

Theorem 3.6. *If G is paradoxical and acts on X without nontrivial fixed points, then X is G-paradoxical.*

Proof. Suppose $A_i, B_j \subseteq G$, $g_i, h_j \in G$ witness that G is paradoxical. By the Axiom of Choice, there is a set M containing exactly one element from each G-orbit in X. Then $\{g(M) : g \in G\}$ is a partition of X by Lemma 3.5. Now, let $A_i^* = \bigcup\{g(M) : g \in A_i\}$ and $B_j^* = \bigcup\{g(M) : g \in B_j\}$. Then $\{A_i^*\} \cup \{B_j^*\}$ is a pairwise disjoint collection of subsets of X (because $\{A_i\} \cup \{B_j\}$ is pairwise disjoint) and the equations $X = \bigcup g_i A_i^* = \bigcup h_j B_j^*$ follow from the corresponding equations in G, where $G = \bigcup g_i A_i = \bigcup h_j B_j$. $\square$

As a special case of the previous theorem, we reach the Hausdorff Paradox. As we are dealing with rotations of a sphere in $\mathbb{R}^3$, the subgroup F does not act on S^2 without nontrivial fixed points since each rotation leaves two points fixed, as illustrated in Figure 2. In order to apply Theorem 3.6, we would need to discount these points. It turns out that this is exactly what is done to complete the proof, and so the paradoxical decomposition yields a partition on S^2 minus the set of nontrivial fixed points of F.

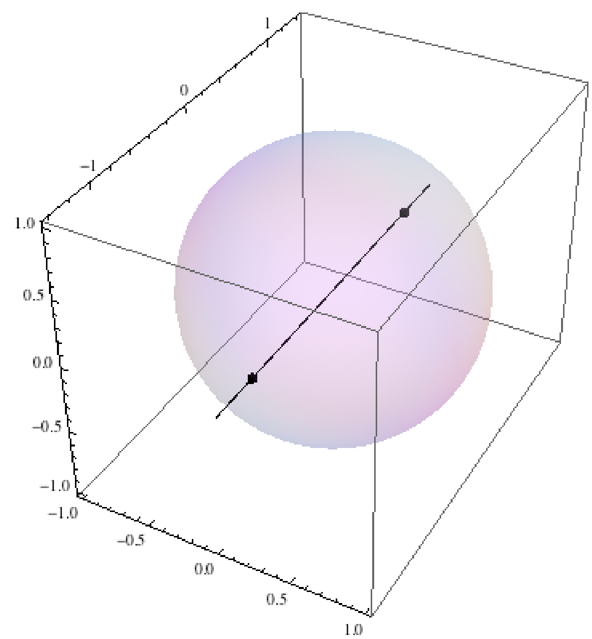


Figure 2: Fixed points of a rotation in S^2

Theorem 3.7. (Hausdorff Paradox) *There is a countable subset D of S^2 such that $S^2\backslash D$ is SO_3-paradoxical.*

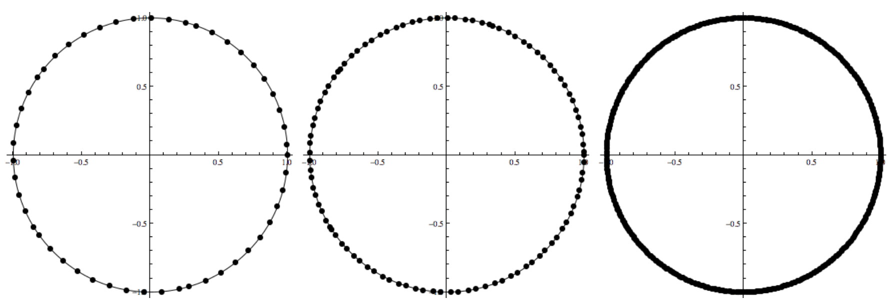
Proof. Let F be the subgroup of Proposition 2.5. For each nontrivial element in F, the axis of rotation intersects S^2 at two points, which are the fixed points of that rotation. Consider the collection D of fixed points of nontrivial rotations in F. Since F is countable, then the set of fixed points is also.

Now, consider $P \in S^2\backslash D$ and $\rho \in F$. Then $\rho(P) \in S^2\backslash D$ since if $\psi \in F$ fixed $\rho(P)$, P would be a fixed point of $\rho^{-1}\psi\rho$, which is a contradiction since $P \notin D$. Thus, F acts on $S^2\backslash D$ without nontrivial fixed points. By Theorem 3.6, $S^2\backslash D$ is F-paradoxical since F is by Theorem 2.6, and since F is a subgroup of SO_3, we have that $S^2\backslash D$ is SO_3-paradoxical. $\square$

To further illustrate this decomposition, and especially the partition created by way of the choice set, M, in Theorem 3.6, we will examine the action of F on $S^2 \backslash D$ that yields the paradoxical result.

Example 3.8. *The subgroup F in Proposition 2.5 admits a paradoxical decomposition of $S^2 \backslash D$, where D is the set of nontrivial fixed points in the subgroup.*

Consider the set $S^2 \backslash D$, where D is the countable set of nontrivial fixed points of F. By Proposition 3.4, the set of F-orbits forms a partition of $S^2 \backslash D$. We now use the Axiom of Choice to choose a single element from each of these orbits to form a choice set, M. Note that the number of F-orbits is uncountable since $S^2 \backslash D$ is, and since each orbit contains only a countable number of elements (corresponding to each of the countable rotations in F). Thus, the choice set M is uncountable. Furthermore, since the choice set is created arbitrarily, the points may be scattered all across $S^2 \backslash D$. However, it is interesting to note that given an open connected region A on S^2, we can choose M such that $M \subset A$, where A is as small as we'd like. The reason for this is that each orbit is countably dense on the sphere, by an application of Kronecker's Approximation Theorem. Figure 3 provides an illustration to show how successive iterations of $\arccos(\frac{1}{3})$ "fill in" the unit circle. With this, we are able to get as close as we'd like to any given point on the sphere by using rotations around the x- and z-axis as a type of coordinate system, rotating around the x-axis and then the z-axis appropriately.

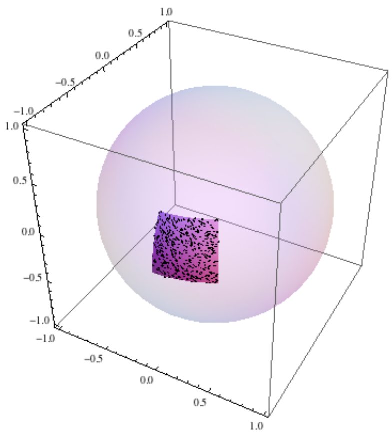


(a) n = 50 (b) n = 100 (c) n = 500

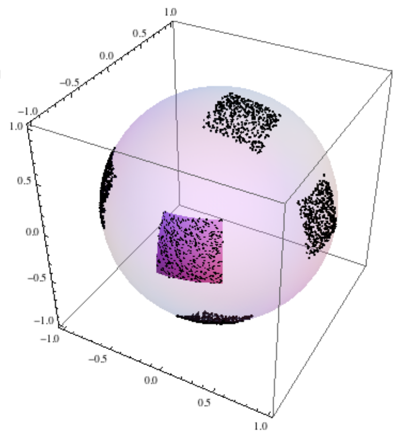
Figure 3: Successive iterations, n, of rotating $(0, 1)$ by $\arccos \frac{1}{3}$

Say that A is the region pictured in Figure 4 (a), and that the uncountable points of M are chosen so that they fall within the region A. Now, following the proof of Theorem 3.6, we create another partition of $S^2 \backslash D$ by rotating M by each element of F. That is, we consider the collection of sets given by $\{\rho(M) : \rho \in F\}$ (a few of which are also pictured in Figure 4). We now have a partition of $S^2 \backslash D$ where each partition set corresponds to a unique element in F. Thus, we are ready to create our paradoxical decomposition.

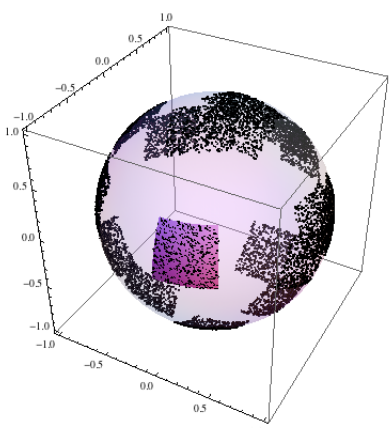
As in the proof of Theorem 2.6, we sort the partitions depending on the element of F with which the corresponding rotation begins. Thus, we will denote



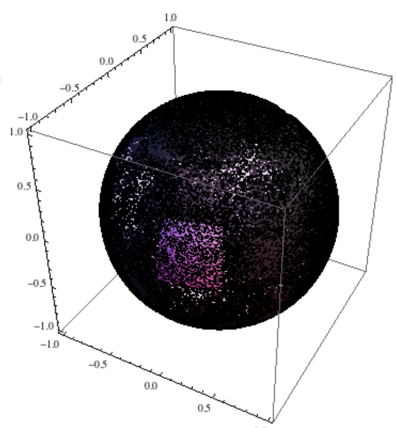
(a) Open region A containing a subset of the choice set M, generated randomly for illustration.



(b) The set representing M rotated by each of the four generating elements of F.



(c) The sets in the previous figure (b) rotated again by each of the four generating elements of F.



(d) Additional rotations by element of F, showing how rotating M under F fills in the sphere, minus a countable set of fixed points.

Figure 4: The choice set M rotated by selected elements of F.

$W^*(\rho) = \bigcup\{\psi(M) : \psi \in W(\rho)\}$, where $W(\rho)$ is defined as in the theorem. Then $\{W^*(\sigma^{\pm 1})\} \cup \{W^*(\tau^{\pm 1})\}$ is a pairwise disjoint collection of subsets of $S^2 \backslash D$ (because $\{W(\sigma^{\pm 1})\} \cup \{W(\tau^{\pm 1})\}$ is pairwise disjoint) and the equations $S^2 \backslash D = W^*(\sigma) \cup \sigma W^*(\sigma^{-1}) = W^*(\tau) \cup \tau W^*(\tau^{-1})$ follow from the corresponding equations in F, where $F = W(\sigma) \cup \sigma W(\sigma^{-1}) = W(\tau) \cup \tau W(\tau^{-1})$.

Thus, to create the paradoxical decomposition, we separated the partition of rotated choice sets into two collections, depending on the element with which the rotations began. Then, collecting together the sets corresponding to rotations beginning with $\sigma^{\pm 1}$, we apply an appropriate permutation and recreate the original set $S^2 \backslash D$. Note that the sets corresponding to rotations beginning with σ remain unchanged. However, the sets corresponding to rotations beginning with σ^{-1} are multiplied on the left by σ, which then results with the collection of partitions obtained by rotating M by elements that begin, on the left, with σ^{-1}, τ, or τ^{-1}. The key is in the unboundedness of the lengths of products in F, much like the paradoxical decomposition of the integers. □

We are now ready to extend the above construction to the entire set S^2.

4 The Banach-Tarski Paradox

To create a paradoxical decomposition of S^2 and not just a subset thereof, we must find a way to fill in the countable set D. To accomplish this, we will introduce the notion of equidecomposability. Techniques involving this definition are used to absorb a countable number of problematic points (such as the fixed points of F), as will be illustrated.

Definition 4.1. Suppose G is a group of permutations of X and $A, B \subseteq X$. Then A and B are *G-equidecomposable* if A and B can be partitioned into the same finite number of respectively G-congruent pieces. In this case we write $A \sim_G B$. Formally,

$$A = \bigcup_{i=1}^{n} A_i, \qquad B = \bigcup_{i=1}^{n} B_i$$

$A_i \cap A_j = \varnothing = B_i \cap B_j$ if $i < j \leq n$, and there are $g_1, ..., g_n \in G$ such that, for each $i \leq n$, $g_i(A_i) = B_i$.

Thus, G-equidecomposability implies that a set A is piecewise congruent to B using permutations from G. In fact, the relation given by $\sim_G$ is an equivalence relation. Using this, we can show that equidecomposability preserves paradoxical decompositions, as the next proposition shows.

Proposition 4.2. *Suppose G is a group of permutations of X and E, E' are G-equidecomposable subsets of X. If E is G-paradoxical, then so is E'.*

Proof. Notice that E is G-paradoxical if and only if E contains disjoint sets A, B such that $A \sim_G E$ and $B \sim_G E$. Thus, by the transitivity of $\sim_G$, we see that $A \sim_G E \sim_G E'$ and $B \sim_G E \sim_G E'$ yield that $A \sim_G E'$ and $B \sim_G E'$ as desired. □

We will now leverage equidecomposability to extend the Hausdorff Paradox to all of S^2. In the proof of the following theorem, we show it is possible to absorb any countable subset D of S^2 using equidecomposability.

Theorem 4.3. *If D is a countable subset of S^2, then S^2 and $S^2 \backslash D$ are SO_3-equidecomposable (using two pieces).*

Proof. We seek to find a rotation, ϕ, of the sphere such that the sets D, $\phi(D)$, $\phi^2(D)$, ... are pairwise disjoint. Denote $\bar{D} = \bigcup\{\phi^n(D) : n = 0, 1, 2, ...\}$. Then since $\bar{D} = \phi^{-1}(\phi(\bar{D}))$ we have $\bar{D} \sim \phi(\bar{D})$. Thus, we would then have $S^2 = \bar{D} \cup (S^2 \backslash \bar{D}) \sim \phi(\bar{D}) \cup (S^2 \backslash \bar{D}) = S^2 \backslash D$. Let ℓ be a line through the origin that misses the countable set D (this is possible since $S^2 \backslash D$ is uncountable, and so if for every point p in $S^2 \backslash D$ the point on S^2 collinear to p and the origin fell in D, then D would have to be uncountable which is a contradiction). Let A be the set of angles θ such that for some $n > 0$ and some $P \in D$, $\psi(P)$ is also in D where ψ is the rotation about ℓ through $n\theta$ radians. Then A is countable (since the set of angles in A corresponds to the countable points in D and the set of natural numbers), so we may choose an angle θ not in A; let ϕ be the corresponding rotation about ℓ. Then $\phi^n(D) \cap D = \varnothing$ if $n > 0$, from which it follows that whenever $0 \leq m < n$, then $\phi^m(D) \cap \phi^n(D) = \varnothing$ (consider $\phi^{n-m}(D) \cap D$); therefore ϕ is as required. □

In order to get a better intuition for how the countable fixed point set D in the Hausdorff Paradox is absorbed, we will continue where we left of in Example 3.8, examining how $S^2 \backslash D \sim S^2$.

Example 4.4. *The set $S^2 \backslash D$, where D is the set of fixed points in the subgroup F in Theorem 2.6, is SO_3-equidecomposable to all of S^2 using two pieces.*

Unfortunately we are unable to constructively provide an example of how Theorem 4.3 applies to our running example. However, using the techniques presented in the proof, we are able to show that such an equivalence is possible in a non-constructive way by proving the existence of a line ℓ that passes through the origin along with two points in $S^2 \backslash D$ and a rotation, ϕ, under which successive iterations generate pairwise disjoint sets when applied to the set D. For the rest of this example, consider such a line ℓ and a rotation ϕ.

We now examine the equidecomposability of $S^2 \backslash D$ and S^2. Intuitively, thinking back to the $(1, 0)$-orbit generated by applying rotations of the form $n \arccos(\frac{1}{3})$ about the unit circle, we can see that applying a single rotation of $-\arccos(\frac{1}{3})$ to the orbit shifts every point backwards by one iteration. Had we removed any point, such as the point $(1, 0)$ itself, a single application of the rotation $-\arccos(\frac{1}{3})$ would get us back to the original set. Essentially this is what we will do on the surface of the sphere to fill in the set D.

The trick is to look not at D itself, but at $\bar{D}$, the set of all rotations of D about ℓ of the form $\phi^n(D)$. That is, the orbit of D under the set of rotations about ℓ of the form ϕ^n. By pairwise disjointness, it follows that for $n > 1$, $D \cap \phi^n(D) = \varnothing$, and thus $\phi^n(D) \subset S^2 \backslash D$ for any such n. Now, the relation $\phi(\bar{D}) \sim \bar{D}$ holds by the application of ϕ^{-1} to the left side of the equivalence.

Furthermore, when ϕ^{-1} is applied to the set $\phi(\bar{D})$, the countable set D is filled in by the rotation. Thus, as in the proof of Theorem 4.3, we have that $S^2 \backslash D \sim S^2$. When looked at from the opposite direction, where the rotation ϕ is applied to D instead of applying ϕ^{-1} to $\phi(\bar{D})$, it is easy to see why this is referred to as the absorption property of equidecomposability. □

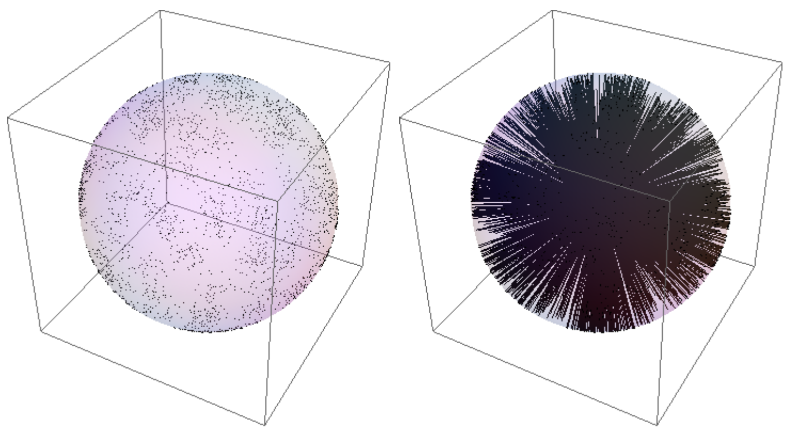
Corollary 4.5. *S^2 is SO_3-paradoxical, as is any sphere centered at the origin.*

Proof. The Hausdorff Paradox (3.7) states that $S^2 \backslash D$ is SO_3-paradoxical for some countable set D (of fixed points of rotations). Combining this with the previous theorem and Proposition 4.2 yields that S^2 is SO_3-paradoxical. None of the previous results depend on the size of the radius of the sphere, and so the result holds for any sphere centered at the origin. □

Corollary 4.6. (The Banach-Tarski Paradox) *The solid unit ball centered at the origin in $\mathbb{R}^3$ is paradoxical under the group of isometries on $\mathbb{R}^3$. Moreover, any solid ball in R^3 is paradoxical with respect to the group of isometries on $\mathbb{R}^3$.*

Proof. Denote the solid unit ball centered at the origin as B. The decomposition of S^2 in the previous corollary admits a similar decomposition for $B \backslash \{\mathbf{0}\}$ using the radial correspondence: $P \to \{\alpha P : 0 < \alpha \leq 1\}$ (see Figure 5). Indeed, the pieces comprised of points P are extended to include all points αP, which essentially each lie on a sphere of radius α. Hence, by showing that B is equidecomposible to $B \backslash \mathbf{0}$ with respect to the group of all isometries of $\mathbb{R}^3$, that is, that the point at the origin can be absorbed, we will achieve the result. Let $P = (0, 0, \frac{1}{2})$ and let ϕ be a rotation of infinite order about an axis through P but missing the origin. Then, the set $D = \{\phi^n(\mathbf{0}) : n \geq 0\}$ may be used to absorb $\mathbf{0}$ as follows: $\phi(D) = D \backslash \{\mathbf{0}\}$, so that $B \sim B \backslash \{\mathbf{0}\}$.

Since the result did not depend on the size of the radius of the ball except for choosing a point on the interior (in this case the point $(0, 0, \frac{1}{2})$), the result holds for arbitrary balls centered at the origin. Additionally, as the group of isometries on $\mathbb{R}^3$ includes all translations, the balls need not be centered at the origin. Hence, the result holds for arbitrary solid balls in $\mathbb{R}^3$. □



(a) Representation of a subset used in the paradoxical decomposition of S^2.

(b) The subset scaled from the surface of S^2 to the origin.

Figure 5: The sets used in the paradoxical decomposition of the solid ball in $\mathbb{R}^3$.